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On the selection of poles in the single-input pole placement problem

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To Hans Schneider for his stable dedication to linear algebra

Abstract

It is well known that the single-input pole placement problem can be quite sensitive to perturbations in the data. Recent results by Mehrmann and Xu show how this sensitivity depends on the location of the poles. In many applications it suffices to prescribe a set **K** in the complex plane that contains the poles. Mehrmann and Xu formulated a minimization problem for allocating the poles in a given set **K** so that the sensitivity to perturbations is reduced. The present paper uses methods of potential theory to derive simple algorithms that yield approximate solutions of this minimization problem. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Consider the single-input time-invariant linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + bu(t), \quad x(0) = x_0, \;\; t \geqslant 0, \tag{1.1}$$

where $A \in \mathbf{C}^{n\times n}$, $b \in \mathbf{C}^n$, $x(t)$ is a vector-valued function with values in $\mathbf{C}^n$ and $u(t)$ is a complex-valued function. The pole placement problem for the system (1.1) has received considerable attention; see, e.g., [1,6,10,13–15] and references therein. It can be formulated as follows: Given a set of n complex numbers $\mathbf{P} = \{\psi_j\}_{j=1}^n$, referred to as poles, find a vector $f \in \mathbf{C}^n$, such that the spectrum of $A - bf^{\mathrm{T}}$, denoted by $sp(A - bf^{\mathrm{T}})$, is $\mathbf{P}$. The vector f is referred to as the feedback gain vector, since if $u(t) := -f^{\mathrm{T}}x(t)$, then (1.1) is a closed-loop system with solution

$$x(t) := \exp((A - bf^{\mathrm{T}})t)x_0, \quad t \geqslant 0.$$

Recently, Mehrmann and Xu [14,16] investigated the conditioning of the single-input pole placement problem. We introduce notation necessary to review some of their results. The matrix–vector pair $\{A, b\}$ is controllable if and only if

$$\mathrm{rank}([b, A - zI]) = n \quad \forall z \in \mathbf{C}.$$

The feedback gain vector f exists for all sets $\mathbf{P} \subset \mathbf{C}$ of distinct poles ψ_j if and only if $\{A, b\}$ is controllable. The distance to uncontrollability is defined as

$$d_{\mathrm{uc}}(A, b) := \min_{z\in\mathbf{C}} \sigma_n([b, A - zI]), \tag{1.2}$$

where $\sigma_n([b, A - zI])$ denotes the nth singular value for the matrix $[b, A - zI]$ and the singular values are ordered in decreasing order.

Let $\|\cdot\|$ denote the Euclidean vector norm as well as the associated induced matrix norm. For a diagonalizable matrix $A \in \mathbf{C}^{n\times n}$ with spectral factorization

$$A = S\Lambda S^{-1}, \qquad \Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n], \tag{1.3}$$

we define the spectral condition number of A as

$$\kappa_A(S) := \|S\|\|S^{-1}\|.$$

Sometimes we denote this condition number simply by $\kappa(S)$. Note that $\kappa_A(S)$ depends on the column scaling of the eigenvector matrix S. Demmel [7] showed that when A has distinct eigenvalues, the condition number $\kappa(S)$ is near-minimal when the columns of S are scaled to have norm one.

Theorem 1.1. *Consider the single-input pole placement problem with data $A \in \mathbf{C}^{n\times n}$, $b \in \mathbf{C}^n$ and $\mathbf{P} = \{\psi_j\}_{j=1}^n$. Assume that $\{A, b\}$ is controllable and that the poles ψ_j are distinct. Let f be the feedback gain of this pole placement* *problem. Also, consider the single-input pole placement problem with perturbed data* $\tilde{A} := A + \delta A$, $\tilde{b} := b + \delta b$ *and* $\mathbf{P}$. *Let*

$$\epsilon := \max\{\|\delta A\|, \|\delta b\|\} \tag{1.4}$$

and assume that

$$2\epsilon < \sigma_\lambda := \min_{1 \leqslant j \leqslant n} \sigma_n([b, A - \psi_j I]). \tag{1.5}$$

Then $\{\tilde{A}, \tilde{b}\}$ *is controllable, and we let* $\tilde{f} := f + \delta f$ *be the feedback gain of the pole placement problem with perturbed data. Let* G_0 *denote the eigenvector matrix of* $A - bf^{\mathrm{T}}$ *with columns normalized to have unit norm, and assume that the constant*

$$c := \epsilon\kappa(G_0)\frac{2\sqrt{2n}}{\sigma_\lambda}\max_{1 \leqslant j \leqslant n}\left(\frac{\|A - \psi_j I\|^2}{\|b\|^2} + 1\right)^{1/2} \tag{1.6}$$

satisfies $c < 1$. *Then*

$$\|\delta f\| \leqslant \frac{c}{1 - c^2}\left(c\|f\| + \sqrt{1 + \|f\|^2}\right). \tag{1.7}$$

Moreover, for each eigenvalue μ_j *of the closed loop matrix* $A - b\tilde{f}^{\mathrm{T}}$, *we have*

$$\min_{1 \leqslant k \leqslant n} \left|\psi_k - \mu_j\right| \leqslant \kappa(G_0)\|b\|\|\delta f\|. \tag{1.8}$$

Proof. Related bounds have been shown by Mehrmann and Xu [14,16]. The proof is similar to the proofs of Theorems 3.2 and 3.3 in [14]. We use closely related notation, and only point out some differences in the proofs.

We first note that the controllability of $\{\tilde{A}, \tilde{b}\}$ follows from (1.4) and (1.5). Let the vectors $w_j \in \mathbf{C}^{n+1}$ be of unit norm and satisfy $[b, A - \lambda_j I]w_j = 0$, $1 \leqslant j \leqslant n$, and introduce the matrix $W = [w_1, w_2, \ldots, w_n] \in \mathbf{C}^{(n+1)\times n}$. Similarly, let the vectors $\tilde{w}_j \in \mathbf{C}^{n+1}$ be of unit norm and satisfy $[\tilde{b}, \tilde{A} - \lambda_j I]\tilde{w}_j = 0$, $1 \leqslant j \leqslant n$, and define $\widetilde{W} = [\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n]$. Let G be the trailing $n \times n$ submatrix of W. Then

$$\delta f^{\mathrm{T}} = \tilde{f}^{\mathrm{T}} - f^{\mathrm{T}} = [1, \tilde{f}^{\mathrm{T}}](W - \widetilde{W})G^{-1}. \tag{1.9}$$

This expression differs from the one used in [14]. Bounds derived by Mehrmann and Xu [14] can be used to obtain

$$\|\delta f\| \leqslant \epsilon(1 + \|\tilde{f}\|^2)^{1/2}\frac{2\sqrt{2n}}{\sigma_\lambda}\|G^{-1}\| \tag{1.10}$$

from (1.9). Let $\widetilde{G}$ be the trailing $n \times n$ submatrix of $\widetilde{W}$. Mehrmann and Xu [14] derived a bound for $\|\widetilde{G}^{-1}\|$, and a bound for $\|G^{-1}\|$ can be obtained in a similar fashion. Substituting this bound into (1.10) yields

$$\|\delta f\| \leqslant c(1+\|\tilde{f}\|^2)^{1/2}, \tag{1.11}$$

where c is given by (1.6). Finally, substituting $\|\tilde{f}\| \leqslant \|f\| + \|\delta f\|$ into (1.11) and solving for $\|\delta f\|$ yields (1.7). Formula (1.8) follows from the Bauer–Fike theorem applied to the matrix $A - bf^{\mathrm{T}}$ with perturbation $b\delta f^{\mathrm{T}}$. □

In many applications it is not necessary to specify the poles ψ_j. Instead, it suffices to prescribe a compact set $\mathbf{K} \subset \mathbf{C}$ that contains all the poles ψ_j. We refer to $\mathbf{K}$ as the candidate pole set. It is desirable that the bound (1.8) for the eigenvalues μ_j of $A - b\tilde{f}^{\mathrm{T}}$, as well as the bound (1.7) for $\|\delta f\|$, be small. Both bounds grow with the condition number $\kappa(G_0)$. We will see below that the condition number $\kappa(G_0)$ can vary considerably with the distribution of the poles. We remark that the quantity σ_λ in (1.5) is bounded below by the distance to uncontrollability (1.2), which is independent of the choice of poles. We are therefore lead to study the minimization problem

$$\min_{\{\psi_j\}_{j=1}^n \subset \mathbf{K}} \kappa(G), \tag{1.12}$$

where G denotes the eigenvector matrix of $A - bf^{\mathrm{T}}$ with a convenient column scaling. A related minimization problem has been formulated by Mehrmann and Xu [16].

The numerical solution of the problem (1.12) is a fairly difficult task already for a small number of poles. We show in Section 2 that the problem (1.12) is related to an interpolation problem for rational functions with fixed poles, and we apply methods of potential theory to derive simple algorithms that yield approximate solutions of (1.12). The special case when A and b have real entries is discussed in Section 3. Computed examples are presented in Section 4.

2. Allocation of poles

Let

$$\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \tag{2.1}$$

with $\lambda_j \neq \lambda_k$ for $j \neq k$, and define

$$e := [1, 1, \ldots, 1]^{\mathrm{T}}. \tag{2.2}$$

Then $\{\Lambda, e\}$ is controllable. Let $\mathbf{P} = \{\psi_j\}_{j=1}^n$ be a given set of distinct poles, and assume that $sp(\Lambda) \cap \mathbf{P} = \emptyset$. Mayne and Murdoch [13] observed that the vector $f_e \in \mathbf{C}^n$, such that

$$sp(\Lambda - ef_e^{\mathrm{T}}) = \mathbf{P}$$

is given by

$$f_e := C^{-\mathrm{T}} e, \tag{2.3}$$

where $C = [c_{jk}]_{j,k=1}^{n}$ is a Cauchy matrix with entries

$$c_{jk} := \frac{1}{\lambda_j - \psi_k}. \tag{2.4}$$

This can be seen as follows. Let e_j denote the jth axis vector. Then

$$e_j^{\mathrm{T}}(\Lambda - ef^{\mathrm{T}})Ce_k = e_j^{\mathrm{T}}(\Lambda C - ee^{\mathrm{T}})e_k = \frac{\lambda_j}{\lambda_j - \psi_k} - 1 = \frac{\psi_k}{\lambda_j - \psi_k}.$$

Therefore

$$(\Lambda - ef_e^{\mathrm{T}})C = C\,\mathrm{diag}[\psi_1, \psi_2, \ldots, \psi_n], \tag{2.5}$$

and it follows that the spectral condition number of $\Lambda - ef_e^{\mathrm{T}}$ is $\kappa(C)$. This result is also discussed by Mehrmann and Xu [14,16].

Following Mehrmann and Xu [16], we note that when $A \in \mathbf{C}^{n\times n}$ is a diagonalizable matrix with spectral factorization (1.3) and distinct eigenvalues λ_j, and $b \in \mathbf{C}^n$ is an arbitrary vector such that no component of $S^{-1}b$ vanishes, the feedback gain vector is given by

$$f^{\mathrm{T}} = f_e^{\mathrm{T}} D_{S^{-1}b}^{-1} S^{-1}, \tag{2.6}$$

and the analogue of (2.5) is

$$(A - bf^{\mathrm{T}})SD_{S^{-1}b}C = SD_{S^{-1}b}C\,\mathrm{diag}[\psi_1, \psi_2, \ldots, \psi_n]. \tag{2.7}$$

Here $D_{S^{-1}b}$ denotes the diagonal matrix whose nontrivial entries are the components of the vector $S^{-1}b$. This vector is related to the distance to uncontrollability. With S scaled so that $\|S^{-1}\| \geqslant 1$, we obtain

$$d_{\mathrm{uc}}(A, b) \leqslant \kappa(SD_{S^{-1}b}C) \min_{1\leqslant j\leqslant n} |(S^{-1}b)_j|,$$

where $(S^{-1}b)_j := e_j^{\mathrm{T}} S^{-1}b$. In particular, the existence of $D_{S^{-1}b}^{-1}$ is equivalent to the controllability of $\{A, b\}$.

Formula (2.7) yields the bound

$$\kappa(SD_{S^{-1}b}C) \leqslant \kappa(C)\kappa(S) \cdot \frac{\max_{1\leqslant j\leqslant n} |(S^{-1}b)_j|}{\min_{1\leqslant j\leqslant n} |(S^{-1}b)_j|}, \tag{2.8}$$

in which only the factor $\kappa(C)$ depends on the poles. The bound (2.8) can be reduced by moving the poles ψ_j so that $\kappa(C)$ is reduced.

We present a bound for the condition number $\kappa(C)$. An exact expression for $\kappa(C)$ based on the Frobenius matrix norm is given by Mehrmann and Xu [16].

Our bound is better suited for the analysis below. The quantity $\mathbf{E}$ denotes the set of eigenvalues $\{\lambda_j\}_{j=1}^n$.

Theorem 2.1. *Let the Cauchy matrix* $C = [c_{jk}]_{j,k=1}^n$ *be defined by* (2.4). *Then*

$$\begin{aligned}\kappa(C) \leqslant c^2 n^2 \max_{1\leqslant j\leqslant n}&\left(|\lambda_j-\psi_j|\prod_{\substack{k=1\\k\neq j}}^n\left|\frac{\lambda_j-\psi_k}{\lambda_j-\lambda_k}\right|\right)\\ &\times\max_{1\leqslant j\leqslant n}\left(|\psi_j-\lambda_j|\prod_{\substack{k=1\\k\neq j}}^n\left|\frac{\psi_j-\lambda_k}{\psi_j-\psi_k}\right|\right),\end{aligned} \tag{2.9}$$

where $c := 1/\min_{\lambda\in\mathbf{E},\psi\in\mathbf{P}}|\lambda-\psi|$.

Proof. It is well known that

$$C^{-1} = -D^{(1)}C^{\mathrm{T}}D^{(2)}, \tag{2.10}$$

where

$$D^{(j)} = \mathrm{diag}[d_1^{(j)}, d_2^{(j)}, \ldots, d_n^{(j)}], \quad j = 1, 2,$$

and

$$\begin{aligned} d_k^{(1)} &:= (\psi_k-\lambda_k)\prod_{\substack{j=1\\j\neq k}}^n\frac{\psi_k-\lambda_j}{\psi_k-\psi_j},\\ d_k^{(2)} &:= (\lambda_k-\psi_k)\prod_{\substack{j=1\\j\neq k}}^n\frac{\lambda_k-\psi_j}{\lambda_k-\lambda_j};\end{aligned} \tag{2.11}$$

see, e.g., [4,8]. The bound (2.9) follows immediately from (2.10), the fact that

$$\kappa(C) = \|C\|\|C^{-1}\| \leqslant \|C\|\|C\|^{\mathrm{T}}\|D^{(1)}\|\|D^{(2)}\|$$

and

$$\|C^{\mathrm{T}}\| = \|C\| \leqslant \|C\|_{\mathrm{F}} = \left(\sum_{j=1}^n\sum_{k=1}^n\frac{1}{|\lambda_j-\psi_k|^2}\right)^{1/2} \leqslant \frac{n}{\min_{\psi\in\mathbf{P}\,\lambda\in\mathbf{E}}|\lambda-\psi|},$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius matrix norm. □

Theorem 2.1 suggests that the minimization problems

$$\min_{\{\psi_l\}_{l=1}^n \subset \mathbf{K}} \max_{1 \leqslant j \leqslant n} \left(|\lambda_j - \psi_j| \cdot \prod_{\substack{k=1 \\ k \neq j}}^{n} \left| \frac{\lambda_j - \psi_k}{\lambda_j - \lambda_k} \right| \right) \tag{2.12}$$

and

$$\min_{\{\psi_l\}_{l=1}^n \subset \mathbf{K}} \max_{1 \leqslant j \leqslant n} \left(|\psi_j - \lambda_j| \cdot \prod_{\substack{k=1 \\ k \neq j}}^{n} \left| \frac{\psi_j - \lambda_k}{\psi_j - \psi_k} \right| \right) \tag{2.13}$$

be studied, where $\mathbf{K}$ denotes a compact set in the complex plane in which the poles are allowed to lie. We note that the poles ψ_j enter both the numerators and denominators of the products in (2.13), but only the numerators of the products in (2.12). This suggests that the products in (2.13) vary more with the distribution of the poles than the products (2.12). We therefore focus on the minimization problem (2.13).

Taking the nth root and the logarithm of the expression in (2.13) yields the equivalent problem of potential theory

$$\min_{\{\psi_l\}_{l=1}^n \subset \mathbf{K}} \max_{1 \leqslant j \leqslant n} \left(\frac{1}{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} \ln \frac{1}{|\psi_j - \psi_k|} - \frac{1}{n} \sum_{k=1}^{n} \ln \frac{1}{|\psi_j - \lambda_k|} \right); \tag{2.14}$$

the sums represent the potential at the points ψ_j generated by positive charges of strength $1/n$ at the points ψ_k and by negative charges of strength $1/n$ at the points λ_k. For notational convenience, we introduce the measure $\mu_{\mathbf{E}}$ defined by

$$\mathrm{d}\mu_{\mathbf{E}}(z) := \frac{1}{n} \sum_{k=1}^{n} \delta(z - \lambda_k),$$

where $\delta(z)$ denotes the Dirac δ-function. Then

$$\frac{1}{n} \sum_{k=1}^{n} \ln \frac{1}{|\psi - \lambda_k|} = \int \ln \frac{1}{|\psi - \zeta|} \mathrm{d}\mu_{\mathbf{E}}(\zeta).$$

The sums on the left-hand side in (2.14) can also be expressed in terms of measures; however, there are n different sums, one for each value of j, and this makes the minimization problem (2.14) difficult to solve. We therefore replace (2.14) by the simpler problem

$$\min_{\mu \in \mathbf{M}_{\mathbf{K}}} \sup_{z \in \mathbf{K}} \left(\int \ln \frac{1}{|z - \zeta|} \mathrm{d}\mu(\zeta) - \int \ln \frac{1}{|z - \zeta|} \mathrm{d}\mu_{\mathbf{E}}(\zeta) \right), \tag{2.15}$$

where $\mathbf{M}_{\mathbf{K}}$ denotes the set of nonnegative Borel measures of total mass one with support in $\mathbf{K}$. We require the set $\mathbf{K}$ to have a connected complement in

$\mathbf{C} \cup \{\infty\}$, be regular in the sense that the Dirichlet problem for the Laplace equation in $(\mathbf{C} \cup \{\infty\}) \setminus \mathbf{K}$ has a solution, and have positive capacity.

Theorem 2.2. *Under the conditions on the set* $\mathbf{K}$ *stated above, the minimization problem* (2.15) *has a unique solution* $\hat{\mu} \in \mathbf{M}_{\mathbf{K}}$. *Moreover, there is a constant* c, *such that*

$$\int \ln \frac{1}{|z-\zeta|} \mathrm{d}\hat{\mu}(\zeta) - \int \ln \frac{1}{|z-\zeta|} \mathrm{d}\mu_{\mathbf{E}}(\zeta) = c, \quad z \in \operatorname{supp} \hat{\mu}, \tag{2.16}$$

and

$$J(\hat{\mu}) \leqslant J(\mu) \quad \forall \mu \in \mathbf{M}_{\mathbf{K}}, \tag{2.17}$$

where

$$J(\mu) := \int \int \ln \frac{1}{|z-\zeta|} \mathrm{d}\mu(\zeta)\, \mathrm{d}\mu(z) - 2 \int \int \ln \frac{1}{|z-\zeta|} \mathrm{d}\mu_{\mathbf{E}}(\zeta)\, \mathrm{d}\mu(z). \tag{2.18}$$

Proof. Theorem 2.2 can be shown under weaker assumptions on the set $\mathbf{K}$; see, e.g., [3] and references therein. A recent discussion can be found in [12]. □

A measure $\hat{\mu}_m$ with only m points of increase that approximates $\hat{\mu}$ can be determined from (2.17) as follows. Let

$$\mathrm{d}\hat{\mu}_m(z) := \frac{1}{m} \sum_{j=1}^{m} \delta(z - \psi_j).$$

Then (2.18) yields

$$J(\hat{\mu}_m) = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{\substack{k=1 \\ k \neq j}}^{m} \ln \frac{1}{|\psi_j - \psi_k|} - \frac{2}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} \ln \frac{1}{|\psi_j - \lambda_k|}, \tag{2.19}$$

with the convention that terms of infinite magnitude are excluded from the summation. Assume that $\psi_1, \psi_2, \ldots, \psi_{m-1}$ already have been determined. Then it follows from (2.19) that the problem $\min_{\psi_m \in \mathbf{K}} J(\hat{\mu}_m)$ is equivalent to

$$\min_{\psi_m \in \mathbf{K}} \left(\frac{1}{m} \sum_{j=1}^{m-1} \ln \frac{1}{|\psi_m - \psi_j|} - \frac{1}{n} \sum_{k=1}^{n} \ln \frac{1}{|\psi_m - \lambda_k|} \right). \tag{2.20}$$

Exponentiation transforms (2.20) into

$$\min_{\psi_m \in \mathbf{K}} \frac{\prod_{k=1}^{n} |\psi_m - \lambda_k|^{1/n}}{\prod_{j=1}^{m-1} |\psi_m - \psi_j|^{1/m}}. \tag{2.21}$$

We can use (2.21) to determine $\psi_1, \psi_2, \ldots$ sequentially, where the product in the denominator is one for $m = 1$. The sequence of points ψ_j so obtained is analogous to Leja points [11] for polynomial interpolation. Techniques analogous to those used for Leja points and their generalization known as Bagby points [2] can be used to show that the measure

$$\mathrm{d}\hat{\mu}_m(z) := \frac{1}{m} \sum_{j=1}^{m} \delta(z - \psi_j)$$

defined by the points ψ_j computed by (2.21) converges to $\mathrm{d}\hat{\mu}$ weakly as $m \to \infty$. We remark that the minimization problems (2.17) and (2.21) also arise in the context of interpolation by rational functions with given singularities (poles) at the points λ_k. In this context the ψ_j are interpolation points, whose distribution in a set $\mathbf{K}$ is to be determined; see [2,12].

In summary, an approximate solution to the minimization problems (1.12) can be determined by computing an approximate solution to (2.13). Such a solution is obtained by solving (2.21) for $m = 1, 2, \ldots, n$. Let the set $\mathbf{K}$ consist of L distinct points. Then the poles $\{\psi_j\}_{j=1}^n$ can be computed in this manner in $\mathrm{O}(L(n + m))$ arithmetic floating point operations. In the computed examples of Section 4, we replace sets $\mathbf{K}$ with infinitely many points by discrete sets of finite cardinality in order to simplify the computations.

3. Allocation of real and complex conjugate poles

This section considers the case when the entries of the matrix A and vector b are real. We would like to allocate the poles ψ_j so that the gain vector f has real entries. The following theorem shows how to achieve this.

Theorem 3.1. *Let $A \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^n$ and assume that $\{A, b\}$ is controllable. Assume that the poles ψ_j are distinct, and are real or appear in complex conjugate pairs. Then $f \in \mathbf{R}^n$.*

Proof. For definiteness, assume that the poles are ordered so that

$$\begin{aligned} &\psi_{2j} = \overline{\psi}_{2j-1}, \quad \mathrm{Im}(\psi_{2j-1}) > 0, \qquad && 1 \leqslant j \leqslant r, \\ &\psi_j \in \mathbf{R}, && 2r < j \leqslant n, \end{aligned}$$

where $\overline{\psi}_{2j-1}$ denotes the complex conjugate of ψ_{2j-1}. Let A have spectral factorization (1.3) and assume that the eigenvalues have been ordered so that

$$\begin{aligned} &\lambda_{2k} = \overline{\lambda}_{2k-1}, \quad \mathrm{Im}(\lambda_{2k-1}) > 0, \qquad && 1 \leqslant k \leqslant s, \\ &\lambda_k \in \mathbf{R}, && 2s < k \leqslant n. \end{aligned}$$

Then

$$Se_{2k-1} = \overline{S}e_{2k}, \quad e_{2k-1}^{\mathrm{T}}S^{-1} = e_{2k}^{\mathrm{T}}\overline{S}^{-1}, \qquad 1 \leqslant k \leqslant s,$$

and

$$\mathrm{Im}(Se_k) = 0, \quad \mathrm{Im}(e_k^{\mathrm{T}}S^{-1}) = 0^{\mathrm{T}}, \qquad 2s < k \leqslant n.$$

Since b is real, the rows of $S^{-1}b$ display an analogous behaviour. Let $m := \max\{r, s\}$. Then $(f_e)_{2k-1} = (\overline{f_e})_{2k}$ for $1 \leqslant k \leqslant m$. Complex entries appear in conjugate pairs, and therefore $f_j = f_e^{\mathrm{T}}(D_{S^{-1}b}^{-1}S^{-1})e_j$ is a sum of m conjugate pairs and $n - 2m$ real numbers. It follows that the vector f has real entries. □

In view of Theorem 3.1, we would like the allocated poles to be real or appear in complex conjugate pairs when $A \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^n$. In order to achieve this, we derive an analogue of formula (2.21) for the simultaneous allocation of a pole and its complex conjugate. Consider formula (2.19) for $m = 2$, and assume that $\psi_2 = \overline{\psi}_1$. We obtain

$$J(\hat{\mu}_2) = \frac{1}{2}\ln\frac{1}{|2\,\mathrm{Im}(\psi_1)|} - \frac{2}{n}\sum_{k=1}^{n}\ln\frac{1}{|\psi_1 - \lambda_k|}. \tag{3.1}$$

Introduce the set $\mathbf{K}_+ := \{z \in \mathbf{K} : \mathrm{Im}(z) > 0\}$. Minimization of (3.1) over $\psi_1 \in \mathbf{K}_+$ is equivalent to the problem

$$\inf_{\psi_1 \in \mathbf{K}_+} \frac{\prod_{k=1}^{n}|\psi_1 - \lambda_k|^{1/n}}{|\mathrm{Im}(\psi_1)|^{1/4}}. \tag{3.2}$$

The solution of (3.2) yields the poles ψ_1 and $\psi_2 := \overline{\psi}_1$.

Assume that the poles $\psi_1, \psi_2, \ldots, \psi_{m-2}$ have been allocated and that they are real or appear in complex conjugate pairs. We wish to determine $\psi_{m-1} \in \mathbf{K}_+$ and $\psi_m = \overline{\psi}_{m-1}$ so that $J(\hat{\mu}_m)$ is minimized. Analogously to (3.2), we obtain the minimization problem

$$\inf_{\psi_{m-1} \in \mathbf{K}_+} \frac{\prod_{k=1}^{n}|\psi_{m-1} - \lambda_k|^{1/n}}{|\mathrm{Im}(\psi_1)|^{1/2m}\prod_{j=1}^{m-2}|\psi_{m-1} - \psi_j|^{1/m}}, \tag{3.3}$$

which yields ψ_{m-1} and $\psi_m := \overline{\psi}_{m-1}$.

Formulas (3.2) and (3.3) yield pairs of complex conjugate poles. These formulas can be combined with formula (2.21) when some real poles are desired. This is the case when the set of eigenvalues $\{\lambda_k\}_{k=1}^n$ and the set $\mathbf{K}$ are symmetric with respect to the real axis and an odd number of poles are to be allocated.

4. Numerical examples

This section presents several computed examples where, given a controllable pair $\{A, b\}$ and a compact candidate pole set $\mathbf{K} \subset \mathbf{C}$, we solve the pole placement problem as outlined by Algorithm 4.1 below. All computations were carried out on an HP 9000/780 workstation in double precision arithmetic with about 16 significant digits. Candidate pole sets $\mathbf{K}$ with infinitely many points are discretized to simplify the computations.

Algorithm 4.1. *Pole allocation and pole placement.*
1. Compute the spectral factorization (1.3).
2. Determine a suitable allocation of poles in a given compact set $\mathbf{K}$ by (2.21), (3.2) and (3.3) or otherwise.
3. Compute the vector f_e.
4. Compute the LU-factorization of the eigenvector matrix S by Gaussian elimination with partial pivoting.
5. Solve the linear system of equations $Sy = b$ for $y = S^{-1}b$.
6. Solve the linear system $S^{\mathrm{T}}f = D_y^{-1}f_e$ for f.

Steps 3–6 of the algorithm previously have been discussed in [5]. We compute the spectral condition number $\kappa(C)$ for different allocations of poles ψ_j, where the matrices A and C are defined by (2.1) and (2.4), respectively, and the vector e is given by (2.2). Recall that the spectral condition number $\kappa(C)$ is a factor in the bound for the spectral condition number for $A - bf^{\mathrm{T}}$; see (2.8).

We used Algorithm 4.1 because of its simplicity, and because steps 3–6 are fairly inexpensive once steps 1 and 2 have been carried out. However, the pole allocation methods of the present paper can be combined with other algorithms for computing a feedback gain vector as well, such as the method by Miminis and Roth [17].

Let $\tilde{f}$ denote the computed feedback gain vector. We wish to determine how close the eigenvalues of the matrix $A - b\tilde{f}^{\mathrm{T}}$ are to the set of poles $\mathbf{P}$. The difference between the sets $\mathrm{sp}(A - b\tilde{f}^{\mathrm{T}})$ and $\mathbf{P}$ is measured by the following metric, defined for compact sets $\mathbf{F}$ and $\mathbf{G}$ in $\mathbf{C}$

$$\mathrm{d}(\mathbf{F}, \mathbf{G}) := \max\left\{\max_{z\in\mathbf{F}} \mathrm{dist}(z, \mathbf{G}), \max_{\zeta\in\mathbf{G}} \mathrm{dist}(\zeta, \mathbf{F})\right\},$$

where $\mathrm{dist}(z, \mathbf{G}) := \min_{\zeta\in\mathbf{G}} |z - \zeta|$.

Example 4.1. This example illustrates the pole placement problem for a matrix $A \in \mathbf{C}^{11\times 11}$, whose spectrum forms a discrete S in the right half-plane. The candidate pole set $\mathbf{K}$ is an H-shaped set in the left half-plane; see Fig. 1. The eigenvalues are marked by + in the figure.

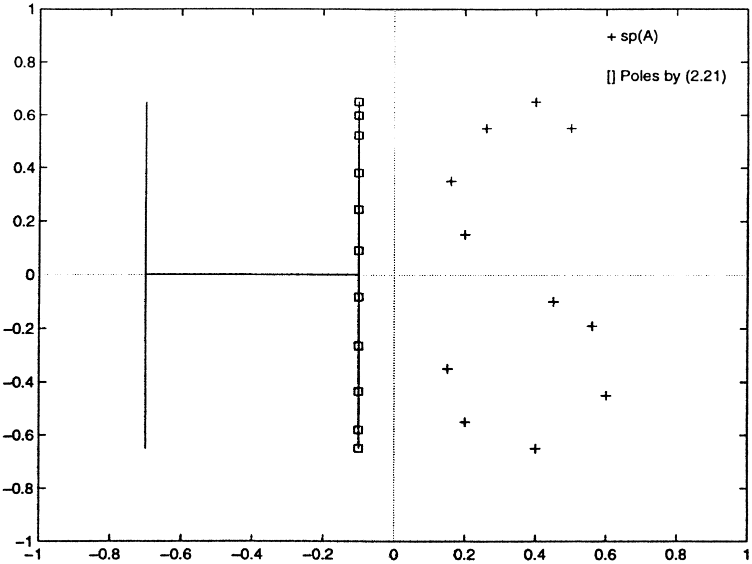

Fig. 1. Example 4.1. Complex plane with 11 poles marked by □ and 11 eigenvalues marked by +. The poles are allocated on the H-shaped candidate pole set using formula (2.21).

Let $b := [1, 1, \ldots, 1]^{\mathrm{T}}$. Since A has complex entries, we only consider method (2.21) for pole allocation. The poles ψ_j determined in this manner are marked by □ in Fig. 1. These poles yield a computed gain vector $\tilde{f}$ of norm 1.9×10^2, and condition number $\kappa^{(\mathrm{F})}(C) = 1.5 \times 10^6$. The superscript (F) indicates that we used the Frobenius norm of C and C^{-1} when evaluating the condition number. The columns of C are scaled to have Euclidean norm one. In particular, the Frobenius norm of $C \in \mathbf{C}^{n \times n}$ is $\sqrt{n}$.

The spectrum of $A - b\tilde{f}^{\mathrm{T}}$ coincides to graphical accuracy with the poles ψ_j; we have $\mathrm{d}(sp(A - b\tilde{f}^{\mathrm{T}}), \mathbf{P}) = 2.6 \times 10^{-10}$.

Example 4.2. Let $A \in \mathbf{R}^{7 \times 7}$ and $b := [1, 0, \ldots, 0]^{\mathrm{T}}$. The eigenvalues of A are marked by + in Fig. 2 and are real or appear in complex conjugate pairs. We would like the poles also to be real or appear in complex conjugate pairs. We let the candidate pole set $\mathbf{K}$ consist of the union of the intervals $[-5.1 + 5\mathrm{i}, -0.1]$ and $[-5.1 - 5\mathrm{i}, -0.1]$. The intervals form a rotated v; see Fig. 2. Sets $\mathbf{K}$ of this form can be attractive when the system (1.1) is to be integrated numerically, because this shape of $\mathbf{K}$ may allow the use of $A(\alpha)$-stable integration methods; see, e.g., [9] for a discussion of such methods.

Fig. 2 displays complex conjugate poles, marked by ∘, allocated by formulas (3.2) and (3.3). Since the number of poles is odd, the first pole (at the vertex of

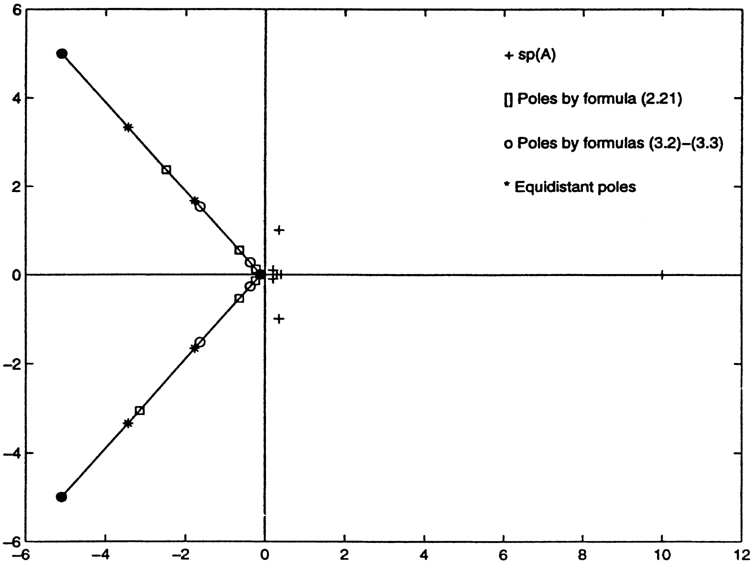

Fig. 2. Example 4.2. Complex plane with seven eigenvalues in the right half-plane marked by $+$ and poles allocated by several methods.

K) is allocated by formula (2.21). The figure also shows poles allocated by other methods, such as by formula (2.21) only. Poles obtained in this manner are not complex conjugate and yield a feedback gain vector with complex entries. We also allocate real or complex conjugate poles equidistantly with respect to arc length on the set **K**. We refer to the poles so obtained as "equidistant poles on **K**." Table 1 summarizes the numerical results for these allocation methods, as well as for a set of poles determined by generating many sets of random poles on **K** and then choosing a set that yields minimal $\kappa(C)$. We refer to the latter method as "Monte Carlo". It is quite expensive.

As one might expect, choosing equidistant poles results in a less well-conditioned problem than when choosing poles by one of the other methods. In

Table 1
Example 4.2. Numerical results

Pole allocation method	$\|\|\tilde{f}\|\|$	$\kappa(C)$	$d(sp(A - b\tilde{f}^{T}), \mathbf{P})$
Formula (2.21)	1.7×10^{2}	1.9×10^{4}	2.6×10^{-12}
Formulas (3.2) and (3.3) for pairs of complex conjugate poles	4.4×10^{4}	1.4×10^{6}	3.6×10^{-11}
Equidistant poles on **K**	2.3×10^{6}	1.6×10^{7}	5.0×10^{-9}
Monte Carlo	2.6×10^{3}	1.8×10^{4}	9.6×10^{-13}

this example, the allocation methods (2.21) and (3.2) and (3.3) yield Cauchy matrices C with quite different condition numbers. Since A has real entries, allocation of complex conjugate poles by formulas (3.2) and (3.3) generally is preferable because it gives a feedback gain vector with real entries.

Example 4.3. This example presents a significantly more ill-conditioned problem than the previous examples. The spectrum of $A \in \mathbf{R}^{11\times 11}$ and the unusual candidate pole set $\mathbf{K}$ are shown in Fig. 3. Let $b := [1, 1, \ldots, 1]^{\mathrm{T}}$. We compare several of the pole allocation strategies that were discussed in Example 4.2. The numerical results are summarized in Table 2. The problem is so poorly conditioned, that the spectrum of $A - b\tilde{f}^{\mathrm{T}}$ is not close to the set $\mathbf{K}$ for the equidistant pole selection. The pole allocation methods based on formulas (2.21) and (3.2) and (3.3) yield computed gain vectors $\tilde{f}$ such that the spectra of the matrices $A - b\tilde{f}^{\mathrm{T}}$ are close to the pole sets $\mathbf{P}$.

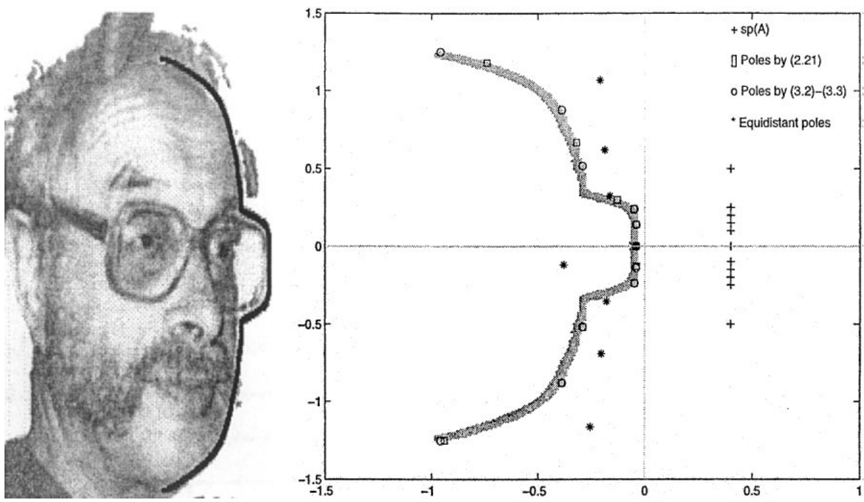

Fig. 3. Example 4.3. Complex plane with 11 eigenvalues along the interval $[\frac{2}{5} - \frac{1}{2}\mathrm{i}, \frac{2}{5} + \frac{1}{2}\mathrm{i}]$ marked by $+$. The candidate pole set $\mathbf{K}$ is the curve in the left-hand side figure.

Table 2
Example 4.3. Numerical results

Pole allocation method	$\lVert\tilde{f}\rVert$	$\kappa(C)$	$\mathrm{d}(sp(A - b\tilde{f}^{\mathrm{T}}), \mathbf{P})$
Formula (2.21)	1.3×10^{6}	4.7×10^{8}	2.4×10^{-3}
Formulas (3.2) and (3.3) for pairs of complex conjugate poles	2.4×10^{6}	6.0×10^{8}	3.3×10^{-3}
Equidistant poles on **K**	5.7×10^{7}	4.9×10^{10}	1.4×10^{0}

5. Conclusion

Pole placement problems can be severely ill-conditioned. However, by choosing the poles judiciously in a prescribed compact set in the complex plane, the condition number often can be reduced. We derive simple methods for determining suitable sets of poles. Numerical examples illustrate that, indeed, the condition number obtained often is smaller than the condition number for pole placement problems with poles selected in an ad hoc manner.

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